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On tensor operators of the Lie superalgebra OSP(1, 2)

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Abstract. We give a complete theory on tensor operators of the OSP(1, 2) algebra, including a new definition of the irreducible tensor, a coupling law of two irreducible tensors, the Wigner-Eckart theorem and a series of methods of calculating reduced matrix elements, etc. Here one can see the overall application of the OSP(1, 2) Racah coefficients.

1. Introduction

In a previous paper [1] we studied the coupling coefficients of the OSP(1, 2) algebra which are the straightforward generalisations of the corresponding coefficients of the SO(3) algebra. In order to extend all the calculating techniques of the SO(3) algebra to the OSP(1, 2) algebra, we study the tensor operators of the OSP(1, 2) algebra in this paper.

Several authors have investigated the tensor operators of Lie superalgebras. Pais and Rittenberg have given a narrower definition for the irreducible tensor [2]. Mezincescu and Agrawala have studied the Wigner-Eckart theorem by integration over the group and algebraic methods, respectively [3, 4]. However, the coupling coefficients have not been studied in these works and the calculation of matrix elements for tensor operators cannot actually be carried out.

We attempt to solve all these questions with respect to the calculation of matrix elements for the tensor operators of the OSP(1, 2) algebra. With this end in view, we first give a new and general definition for the irreducible tensor and then use our results in respect of the coupling laws and coupling coefficients [1] to complete the demonstration of several theorems and to derive all the required relations. The following sections include: the definition of the irreducible tensor, the product of two irreducible tensors, the Wigner-Eckart theorem, the projection theorem on a tensor of rank one, a calculation of the reduced matrix elements in the coupling spaces and the coupling laws for reduced matrix elements, etc. In the expositions of these subjects, we will see the overall applications of CG coefficients and Racah coefficients of the OSP(1, 2) algebra.

2. Definition of the irreducible tensor

In [2], an irreducible tensor (or, more correctly, its components) is denoted by T^J_{IM}, where $I = J, J-\frac{1}{2}$; $M = I, I-1, \ldots, -I$ and T^J_{JM} is assumed to be the even element of the tensor, namely $\lambda(J) = 0$ ($\lambda(I)$ denotes the degree for T^J_{IM}). If an irreducible tensor is constructed via coupling, the assumption $\lambda(J) = 0$ is not possible in general cases.

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Therefore, this definition of the irreducible tensor is narrower; it cannot be used to establish a complete theory of tensor operators.

We will give a new definition for the irreducible tensor. First of all, we relabel the matrix elements of an operator. In order to use an indefinite metric, we write the matrix element of the operator A as $\varepsilon(\psi)\langle\psi|A|\varphi\rangle$ and assume that A always acts on the right vector $|\varphi\rangle$. $\varepsilon(\psi)$ is the norm of the vector $|\psi\rangle$ generated from the action of the operator A on the vector $|\varphi\rangle$.

Next we redenote the generators of the OSP(1, 2) algebra by the unified notation q^2_m $(m=2, 1, 0, -1\ -2)$, namely, let

$$q^2_2=-(2)^{-1/2}Q_+ \qquad q^2_0=\tfrac{1}{2}Q_3 \qquad q^2_{-2}=(2)^{-1/2}Q_- \qquad q^2_1=V_+ \qquad q^2_{-1}=V_- \tag{2.1}$$

and write the matrix element of the generator q^2_m as $\varepsilon(2J, M')\langle 2J, M'|q^2_m|2J, M\rangle$ where $M'=M+m$ and $\varepsilon(2J, M')$ is the norm of the vector $|2J, M'\rangle$ generated from the action of the generator q^2_m on the vector $|2J, M\rangle$.

We denote an irreducible tensor (or, more correctly, its components) by T^{2J}_M with $M=2J, 2J-1, \ldots, -2J$ and denote its degree by $\lambda(M)$. T^{2J}_M may be classified into two parts corresponding to even $(2J-M)$ and odd $(2J-M)$, respectively. Each part has the same degree. The degree of the part of even $(2J-M)$ is $\lambda(2J)$. Our definition of the irreducible tensor is as follows†:

$$\langle q^2_m, T^{2J}_M\rangle=\varepsilon(2J, M')\langle 2J, M'|q^2_m|2J, M\rangle T^{2J}_{M'} \tag{2.2}$$

where the notation $\langle\ ,\ \rangle$ means

$$\langle q^2_m, T^{2J}_M\rangle=q^2_m T^{2J}_M-(-1)^{\lambda(m)\lambda(M)}T^{2J}_M q^2_m \tag{2.3}$$

while $\lambda(m)$ and $\lambda(M)$ are the degrees of q^2_m and T^{2J}_M, respectively.

We have not generally assumed $\lambda(2J)=0$ for T^{2J}_M. In fact, there is no need for us to designate which components of T^{2J}_M are even elements. The designation for $\lambda(2J)$ only has an effect on the phases of the matrix elements for T^{2J}_M (see § 4).

We call J the rank of the irreducible tensor T^{2J}_M. The generators are clearly an irreducible tensor of rank one. Because q^2_m of even $(2-m)$ are the even elements among generators, we must therefore set $\lambda(2)=0$ for generators.

3. Product of two irreducible tensors

Now we consider two irreducible tensors, $S^{2J_1}_{M_1}$ and $T^{2J_2}_{M_2}$, and construct

$$[S^{2J_1}\times T^{2J_2}]^{2J}_M=\sum_{M_1M_2}\begin{pmatrix}2J_1 & 2J_2 & 2J\\ M_1, & M_2; & M\end{pmatrix}S^{2J_1}_{M_1}T^{2J_2}_{M_2}. \tag{3.1}$$

Taking note of

$$\begin{aligned}\langle q^2_m, S^{2J_1}_{M_1}T^{2J_2}_{M_2}\rangle&=\varepsilon(2J_1, M_1')\langle 2J_1, M_1'|q^2_m|2J_1, M_1\rangle S^{2J_1}_{M_1'}T^{2J_2}_{M_2}\\&\quad+(-1)^{\lambda(m)\lambda(M_1)}\varepsilon(2J_2, M_2')\langle 2J_2, M_2'|q^2_m|2J_2, M_2\rangle S^{2J_1}_{M_1}T^{2J_2}_{M_2'}\end{aligned} \tag{3.2}$$

and using the orthogonality relations of CG coefficients, we may show that $[S^{2J_1}\times T^{2J_2}]^{2J}_M$ satisfies

$$\langle q^2_m, [S^{2J_1}\times T^{2J_2}]^{2J}_M\rangle=\varepsilon(2J, M')\langle 2J, M'|q^2_m|2J, M\rangle[S^{2J_1}\times T^{2J_2}]^{SJ}_{M'}. \tag{3.3}$$

† This definition is very similar to that given by Agrawala [4].

namely it is also an irreducible tensor. We call $[S^{2J_1} \times T^{2J_2}]^{2J}_M$ the product of the tensors $S^{2J_1}_{M_1}$ and $T^{2J_2}_{M_2}$.

The relation (3.1) is similar to the coupling law of two irreps [1]. Therefore the degree for $[S^{2J_1} \times T^{2J_2}]^{2J}_M$ is $\lambda(M) = \lambda(M_1) + \lambda(M_2)$, while

$$\lambda(2J) = \lambda(2J_1) + \lambda(2J_2) + 2(J_1 + J_2 - J) \tag{3.4}$$

which is determined up to an even integer.

It is interesting to set $J = 0$ (thus $M = 0$) in (3.1). In order to obtain this particular product, we must set $J_1 = J_2$, $M_1 = -M_2$ and $\lambda(2J_1) = \lambda(2J_2)$. Replacing J_1, M_1 by J, M and using a particular value of CG coefficients

$$\begin{pmatrix} 2J & 2J & 0 \\ M, & -M; & 0 \end{pmatrix} = (-1)^{(M^> - M)/2 + (2J - M)\lambda(M)} \tag{3.5}$$

(see [1] for the definition of $M^>$) we get

$$[S^{2J} \times T^{2J}]^0_0 = \sum_M (-1)^{(M^> - M)/2 + (2J - M)\lambda(M)} S^{2J}_M T^{2J}_{-M}. \tag{3.6}$$

$[S^{2J} \times T^{2J}]^0_0$ is the scalar product of the tensors S^{2J}_M and T^{2J}_M. It is clear that the scalar product is constructed only from two tensors having the same rank and same designation of degree.

Let $S^{2J}_M = T^{2J}_M = q^2_m$ ($J = 1$, $M = m$); then it follows that

$$[q^2 \times q^2]^0_0 = \sum_m (-1)^{(m^> - m)/2 + \lambda(m)} q^2_m q^2_{-m}. \tag{3.7}$$

One easily sees $[q^2 \times q^2]^0_0 = -\frac{1}{4}K$, where K is the Casimir operator of the OSP(1, 2) algebra. Therefore, the Casimir operator is the scalar product of the generators with itself.

4. Wigner–Eckart theorem

From the general definition of the irreducible tensor and the coupling law of irreps, we may give a strict demonstration for the Wigner–Eckart theorem.

We consider the irreducible tensor $T^{2J_2}_{M_2}$ and write its matrix element as $\varepsilon(2J, M)\langle 2J, M|T^{2J_2}_{M_2}|2J_1, M_1\rangle$†, where $M = M_1 + M_2$ and $\varepsilon(2J, M)$ is the norm of the vector $|2J, M\rangle$ generated from the action of $T^{2J_2}_{M_2}$ on the vector $|2J_1, M_1\rangle$. In our version, the Wigner–Eckart theorem associated with the matrix element of $T^{2J_2}_{M_2}$ takes the following form.

Theorem. The matrix element of $T^{2J_2}_{M_2}$ may be expressed in terms of the CG coefficient as follows:

$$\varepsilon(2J, M)\langle 2J, M|T^{2J_2}_{M_2}|2J_1, M_1\rangle = (2J\|T^{2J_2}\|2J_1)(-1)^f \begin{pmatrix} 2J_1 & 2J_2 & 2J \\ M_1, & M_2; & M \end{pmatrix} \tag{4.1}$$

where $(2J\|T^{2J_2}\|2J_1)$ is the reduced matrix element of $T^{2J_2}_{M_2}$, while

$$f = 2(J_1 + J_2 - J)(2J - M) + (2J_1 - M_1)(2J_2 - M_2) + \lambda(M_1)\lambda(M_2).$$

† The notation $|2J, M\rangle$ for an eigenvector suppresses the set of quantum numbers $(\alpha) = (\alpha_1, \alpha_2, \ldots)$ that describe the eigenstates of a set A of observables which, together with K and Q_3, constitute a complete set of commuting observables for a physical system.

In order to prove the theorem, we write down the definition for $T^{2J_2}_{M_2}$:

$$\langle q^2_m, T^{2J_2}_{M_2}\rangle = \varepsilon(2J_2, M'_2)\langle 2J_2, M'_2|q^2_m|2J_2, M_2\rangle T^{2J_2}_{M'_2}. \tag{4.2}$$

Taking the matrix elements of the operators on the two sides of (4.2), using the completeness condition of state vectors with indefinite metrics

$$\sum_M |2J, M\rangle\varepsilon(2J, M)\langle 2J, M| = 1 \tag{4.3}$$

and noting the fact that the sign must be changed if the positions of two odd members are interchanged, we obtain

$$\begin{aligned}(-1)^{\lambda(m)\lambda(M_2)}&\varepsilon(2J_1, M'_1)\langle 2J_1, M'_1|q^2_m|2J_1, M_1\rangle\langle 2J, M|T^{2J_2}_{M_2}|2J_1, M'_1\rangle\\ &+\varepsilon(2J_2, M'_2)\langle 2J_2, M'_2|q^2_m|2J_2, M_2\rangle\langle 2J, M|T^{2J_2}_{M'_2}|2J_1, M_1\rangle\\ &=\langle 2J, M|q^2_m|2J, M'\rangle\varepsilon(2J, M')\langle 2J, M'|T^{2J_2}_{M_2}|2J_1, M_1\rangle.\end{aligned} \tag{4.4}$$

On the other hand, the application of the generator q^2_m on the relation

$$\varepsilon(2J_2, M_2)|2J_2, M_2\rangle\varepsilon(2J_1, M_1)|2J_1, M_1\rangle = \sum_{2J}\begin{pmatrix}2J_1 & 2J_2 & 2J\\ M_1, & M_2; & M\end{pmatrix}\varepsilon(2J, M)|2J, M\rangle \tag{4.5}$$

leads to the result

$$\begin{aligned}(-1)^{\lambda(m)\lambda(M_2)}&\langle 2J_1, M'_1|q^2_m|2J_1, M_1\rangle\varepsilon(2J_1, M_1)\begin{pmatrix}2J_1 & 2J_2 & 2J\\ M'_1, & M_2; & M\end{pmatrix}\\ &+\langle 2J_2, M'_2|q^2_m|2J_2, M_2\rangle\varepsilon(2J_2, M_2)\begin{pmatrix}2J_1 & 2J_2 & 2J\\ M_1, & M'_2; & M\end{pmatrix}\\ &=\langle 2J, M|q^2_m|2J, M'\rangle\varepsilon(2J, M')\begin{pmatrix}2J_1 & 2J_2 & 2J\\ M_1, & M_2; & M'\end{pmatrix}.\end{aligned} \tag{4.6}$$

Comparing the relation (4.4) with (4.6), we find

$$\langle 2J, M|T^{2J_2}_{M_2}|2J_1, M_1\rangle = (2J\|T^{2J_2}\|2J_1)\varepsilon(2J_1, M_1)\varepsilon(2J_2, M_2)\begin{pmatrix}2J_1 & 2J_2 & 2J\\ M_1, & M_2; & M\end{pmatrix} \tag{4.7}$$

where $(2J\|T^{2J_2}\|2J_1)$ is a proportionality factor. Substituting the relation between the metrics of the coupling spaces and that of the in-coupling spaces [1]

$$\varepsilon(2J, M) = (-1)^{2(J_1+J_2-J)(2J-M)+(2J_1-M_1)(2J_2-M_2)+\lambda(M_1)\lambda(M_2)}\varepsilon(2J_1, M_1)\varepsilon(2J_2, M_2) \tag{4.8}$$

into (4.7), we obtain (4.1).

In the Wigner-Eckart theorem (4.1) there is a phase factor $(-1)^f$, which depends on $\lambda(M_1)$, $\lambda(M_2)$ and the odd-even properties of $2(J_1+J_2-J)$, $(2J-M)$, $(2J_1-M_1)$ and $(2J_2-M_2)$ and shows the actions of the degrees of tensor operator and representation spaces.

For example, the matrix elements of the generators may be expressed in terms of CG coefficients as follows:

$$\varepsilon(2J, M')\langle 2J, M'|q^2_m|2J, M\rangle = [J(J+\tfrac{1}{2})]^{1/2}(-1)^{\lambda(2J)\lambda(m)}\begin{pmatrix}2J & 2 & 2J\\ M, & m; & M'\end{pmatrix} \tag{4.9}$$

where we have noted $(2J\|q^2\|2J) = [J(J+\frac{1}{2})]^{1/2}$ and $\lambda(2)=0$ for generators.

If the matrix elements of an irreducible tensor are known, its reduced matrix elements may be obtained from

$$(2J\|T^{2J_2}\|2J_1)=\sum_{M_1M_2}(-1)^{\lambda(M_1)\lambda(M_2)}\varepsilon(2J,M)\langle 2J,M|T^{2J_2}_{M_2}|2J_1,M_1\rangle\begin{pmatrix}2J_1 & 2J_2 & 2J\\ M_1, & M_2; & M\end{pmatrix}. \tag{4.10}$$

5. Projection theorem on a tensor of rank one

According to the Wigner-Eckart theorem, the calculation of a matrix element may be reduced to the calculation of the reduced matrix element.

However, the calculation of the reduced matrix element is not generally a simple matter. Starting from this section, we will give a series of methods for solving this problem.

We first consider a simpler case, namely we calculate the matrix elements of the tensor T^2_m of rank one in the irrep space J. We assume T^2_m has the same designation for degrees as q^2_m, namely $\lambda(2)=0$, and construct the scalar product of T^2_m with q^2_m. Introducing the expansion for $[\boldsymbol{T}^2\times\boldsymbol{q}^2]^0_0$ and using the completeness conditions of state vectors, the Wigner-Eckart theorem and the orthogonality conditions of CG coefficients, we can obtain a relation connecting the matrix element of T^2_m with those of q^2_m:

$$\langle 2J,M'|q^2_m[\boldsymbol{T}^2\times\boldsymbol{q}^2]^0_0|2J,M\rangle=-(2J\|\boldsymbol{q}^2\|2J)^2\langle 2J,M'|T^2_m|2J,M\rangle \tag{5.1}$$

which is called the projection theorem on a tensor of rank one.

Using the Wigner-Eckart theorem again, we further obtain from (5.1)

$$(2J\|\boldsymbol{T}^2\|2J)=-(2J\|\boldsymbol{q}^2\|2J)^{-1}(2J\|[\boldsymbol{T}^2\times\boldsymbol{q}^2]^0\|2J). \tag{5.2}$$

According to the result (5.2), we may calculate the reduced matrix elements of T^2_m from those of q^2_m.

For example, we assume that there are two kinematically independent spaces within which the acting generators are denoted by $q^2_m(1)$ and $q^2_m(2)$, respectively. The reduced matrix element $(2J\|\boldsymbol{q}^2(1)\|2J)$ for $q^2_m(1)$ in the coupling space J can be obtained from (5.2):

$$(2J\|\boldsymbol{q}^2(1)\|2J)=\frac{J(2J+1)+J_1(2J_1+1)-J_2(2J_2+1)}{2[2J(2J+1)]^{1/2}}. \tag{5.3}$$

6. Calculation of reduced matrix elements in coupling spaces

We consider two irreducible tensors $S^{2L_1}_{m_1}(1)$ and $T^{2L_2}_{m_2}(2)$, which are kinematically independent and are labelled by the notations (1) and (2) respectively. We may show that if their reduced matrix elements in the in-coupling spaces—$(2J_1'\|\boldsymbol{S}^{2L_1}(1)\|2J_1)$ and $(2J_2'\|\boldsymbol{T}^{2L_2}(2)\|2J_2)$—are given, then their reduced matrix elements in the coupling spaces—$(2J_1',2J_2',2J'\|\boldsymbol{S}^{2L_1}(1)\|2J_1,2J_2,2J)$ and $(2J_1',2J_2',2J'\|\boldsymbol{T}^{2L_2}(2)\|2J_1,2J_2,2J)$—may be calculated as:

$$(2J_1',2J_2',2J'\|\boldsymbol{S}^{2L_1}(1)\|2J_1,2J_2,2J)=(-1)^{J_1}(2J_1'\|\boldsymbol{S}^{2L_1}(1)\|2J_1)R(J_2J_1J'L_1;JJ_1')\delta_{J_2'J_2} \tag{6.1}$$

$$(2J_1',2J_2',2J'\|\boldsymbol{T}^{2L_2}(2)\|2J_1,2J_2,2J)=(-1)^{J_2}(2J_2'\|\boldsymbol{T}^{2L_2}(2)\|2J_2)R(J_1J_2J'L_2;JJ_2')\delta_{J_1'J_1} \tag{6.2}$$

where $R(\ldots)$ are the OSP(1, 2) Racah coefficients defined in [1], while

$$f_1 = n(JL_1J') + n(J_1L_1J_1') + 2(J_1 + J_1' + J + J')\lambda(2J_2) + [1 + \lambda(2J_1)]\lambda(2J) + [1 + \lambda(2J_1')]\lambda(2J')$$

$$f_2 = 2(J_1 + J_2 + J)2(J + L_2 + J') + 2(J_2 + L_2 + J_2')2(J_1 + J_2' + J')$$

(see [1] for the definition of $n(\ldots)$).

The procedure for deriving the formula (6.1) is as follows. We first write the matrix element $\varepsilon(2J', M')\langle 2J', M'|S^{2L_1}_{m_1}(1)|2J, M\rangle$, next expand $|2J, M\rangle$ and $|2J', M'\rangle$ in terms of $|2J_1, M_1\rangle|2J_2, M_2\rangle$ and $|2J_1', M_1'\rangle|2J_2', M_2'\rangle$, respectively, then apply the Wigner–Eckart theorem to the matrix elements of $S^{2L_1}_{m_1}(1)$, and introduce the Racah coefficient $R(J_2J_1J'L_1; JJ_1')$. We note the following facts at the last step. $\varepsilon(2J', M')$ is the norm of the vector $|2J', M'\rangle$ generated from the action of the tensor $S^{2L_1}_{m_1}(1)$ on the vector $|2J, M\rangle$. Since the vector $|2J, M\rangle$ has been expanded in terms of $|2J_1, M_1\rangle|2J_2, M_2\rangle$, it is also the norm $\varepsilon(2J', M')_J$ of the vector $|2J', M'\rangle_J$ generated via the twofold coupling $2J_1$, $2J_2 \to 2J$; $2J$, $2L_1 \to 2J'$ (the middle state is J). On the other hand, one needs $\varepsilon(2J', M')_{J_1'}$ in the definition for $R(J_2J_1J'L_1; JJ_1')$, which is the norm of the vector $|2J', M'\rangle_{J_1'}$ generated via the twofold coupling $2J_1, 2L_1 \to 2J_1'$; $2J_2, 2J_1' \to 2J'$ (the middle state is J_1'). We must make a transformation which changes $\varepsilon(2J', M')_J$ into $\varepsilon(2J', M')_{J_1'}$, namely

$$\varepsilon(2J', M')_J = (-1)^{2(J_1+J_2+J)2(J+L_1+J')+2(J_1+L_1+J_1')2(J_2+J_1'+J')}\varepsilon(2J', M')_{J_1'}. \tag{6.3}$$

We have a similar derivation for the formula (6.2).

If the vectors $|2J, M\rangle$ and $|2J', M'\rangle$ are all expanded in terms of $|2J_1, M_1\rangle|2J_2, M_2\rangle$, we may show that the reduced matrix elements of $S^{2L_1}_{m_1}(1)$ in the coupling spaces have the following symmetry properties:

$$(2J'\|S^{2L_1}(1)\|2J) = (-1)^K(2J\|S^{2L_1}(1)\|2J') \tag{6.4}$$

where

$$K = n(J_1J_2J) + n(J_1J_2J') + 2(J + J')(2L_1 + 1).$$

We have a similar relation for $T^{2L_2}_{m_2}(2)$.

In order to check the correctness of our results, here we give all the reduced matrix elements for $q^2_m(1)$ in the coupling spaces:

$$(2J+2\|\boldsymbol{q}^2(1)\|2J) = \begin{cases} \left(\dfrac{(J+J_1+J_2+1)(J+J_1-J_2+1)(J-J_1+J_2+1)(-J+J_1+J_2)}{2(2J+1)(2J+2)}\right)^{1/2} & \text{for } 2(J_1+J_2-J) \text{ even} \\ \left(\dfrac{(J+J_1+J_2+\frac{3}{2})(J+J_1-J_2+\frac{1}{2})(J-J_1+J_2+\frac{1}{2})(-J+J_1+J_2-\frac{1}{2})}{2(2J+1)(2J+2)}\right)^{1/2} & \text{for } 2(J_1+J_2-J) \text{ odd} \end{cases} \tag{6.5}$$

$$(2J\|\boldsymbol{q}^2(1)\|2J) = \frac{J(2J+1) + J_1(2J_1+1) - J_2(2J_2+1)}{2[2J(2J+1)]^{1/2}} \tag{6.6}$$

$$(2J+1\|\boldsymbol{q}^2(1)\|2J) = (-1)^{\lambda(2J_2)}(J_1 - J_2)\left(\frac{(J+J_1+J_2+1)(-J+J_1+J_2)}{2J(2J+1)(2J+2)}\right)^{1/2} \quad \text{for } 2(J_1+J_2-J) \text{ even} \tag{6.7}$$

$$(2J-1\|\boldsymbol{q}^2(1)\|2J) = (-1)^{\lambda(2J_2)+1}\frac{(2J_1+2J_2+1)}{2}\left(\frac{(J-J_1+J_2)(J+J_1-J_2)}{(2J-1)2J(2J+1)}\right)^{1/2}$$

$$\text{for } 2(J_1+J_2-J) \text{ even.} \qquad (6.8)$$

(The element $(2J-2\|\boldsymbol{q}^2(1)\|2J)$ could be obtained by means of the symmetry relation (6.4) with $L_1=1$.)

7. General coupling laws for reduced matrix elements

Now we consider the product $[\boldsymbol{S}^{2L_1}\times\boldsymbol{T}^{2L_2}]^{2L}_m$ of two irreducible tensors, $S^{2L_1}_{m_1}$ and $T^{2L_2}_{m_2}$, and calculate its matrix element

$$\varepsilon(2J', M')\langle 2J', M'|[\boldsymbol{S}^{2L_1}\times\boldsymbol{T}^{2L_2}]^{2L}_m|2J, M\rangle$$

where $\varepsilon(2J', M')$ is the norm of the vector $|2J', M'\rangle$ generated from the action of the tensor $[\boldsymbol{S}^{2L_1}\times\boldsymbol{T}^{2L_1}]^{2L}_m$ on the vector $|2J, M\rangle$. Using the completeness conditions of state vectors, the Wigner–Eckart theorem, the orthogonality relations and symmetry properties of CG coefficients and the definition of the Racah coefficient, we can derive the reduced matrix element for $[\boldsymbol{S}^{2L_1}\times\boldsymbol{T}^{2L_2}]^{2L}_m$ as follows:

$$(2J'\|[\boldsymbol{S}^{2L_1}\times\boldsymbol{T}^{2L_2}]^{2L}\|2J) = (-1)^g\sum_{2J''}(2J'\|\boldsymbol{S}^{2L_1}\|2J'')$$
$$\times(2J''\|\boldsymbol{T}^{2L_2}\|2J)R(JL_2J'L_1;\,J''L) \qquad (7.1)$$

where

$$g = n(L_1L_2L)+2(L_1+L_2+L)\lambda(2L)+\lambda(2L_1)\lambda(2L_2).$$

The relation (7.1) is called the coupling law of reduced matrix elements. It indicates that, if the reduced matrix elements for $S^{2L_1}_{m_1}$ and $T^{2L_2}_{m_2}$ are known, then we may obtain the reduced matrix elements for $[\boldsymbol{S}^{2L_1}\times\boldsymbol{T}^{2L_2}]^{2L}_m$ by the aid of Racah coefficients.

It is useful to give a particular case of the coupling law (7.1) in which $L=0$, thus $L_1=L_2$, $\lambda(2L_1)=\lambda(2L_2)$. Replacing L_1 by L anew and using the symmetry properties of Racah coefficients and its particular value

$$R(bbdd;\,0f) = (-1)^{n(bdf)+2(b+d+f)(\lambda(2b)+1)} \qquad (7.2)$$

we can obtain

$$(2J\|[\boldsymbol{S}^{2L}\times\boldsymbol{T}^{2L}]^0\|2J) = \sum_{2J'}(-1)^{g_0}(2J\|\boldsymbol{S}^{2L}\|2J')(2J'\|\boldsymbol{T}^{2L}\|2J) \qquad (7.3)$$

where

$$g_0 = 2L+n(JLJ')+[1+2(J+L+J')]\lambda(2L).$$

We give the following results as examples:

$$\sum_{2J'}(-1)^{n(J_1J')}(2J\|\boldsymbol{q}^2(1)\|2J')(2J'\|\boldsymbol{q}^2(1)\|2J) = -J_1(J_1+\tfrac{1}{2}) \qquad (7.4)$$

$$\sum_{2J'}(-1)^{n(J1J')}(2J\|\boldsymbol{q}^2(1)\|2J')(2J'\|\boldsymbol{q}^2(2)\|2J)$$
$$=\tfrac{1}{4}[J_1(2J_1+1)+J_2(2J_2+1)-J(2J+1)]. \qquad (7.5)$$

Substituting the symmetry relation (6.4) into the result (7.4), it follows that

$$\sum_{2J'}(-1)^{p}(2J'\|\boldsymbol{q}^{2}(1)\|2J)^{2}=J_{1}(J_{1}+\tfrac{1}{2}) \tag{7.6}$$

where

$$p=2(J_{1}+J_{2}+J')2(J+J').$$

8. Coupling laws for reduced matrix elements of tensor operators acting in kinematically independent spaces

We set $S^{2L_1}_{m_1}=S^{2L_1}_{m_1}(1)$ and $T^{2L_2}_{m_2}=T^{2L_2}_{m_2}(2)$. Substituting (6.1) and (6.2) into (7.1), and using the symmetry properties of Racah coefficients, we can derive the coupling laws for reduced matrix elements of two tensor operators acting in kinematically independent spaces, as follows:

$$(2J'_1, 2J'_2, 2J'\|[\boldsymbol{S}^{2L_1}(1)\times\boldsymbol{T}^{2L_2}(2)]^{2L}\|2J_1, 2J_2, 2J)$$
$$=(-1)^{G}(2J'_1\|\boldsymbol{S}^{2L_1}(1)\|2J_1)(2J'_2\|\boldsymbol{T}^{2L_2}(2)\|2J_2)\begin{pmatrix}J_1 & J_2 & J\\ L_1 & L_2 & L\\ J'_1 & J'_2 & J'\end{pmatrix} \tag{8.1}$$

where

$$\begin{pmatrix}J_1 & J_2 & J\\ L_1 & L_2 & L\\ J'_1 & J'_2 & J'\end{pmatrix}=\sum_{J''}(-1)^{n(J_1J''J'_2)}R(J'_2J''J'_1L_1;\, J_1J')R(J_1JJ'_2L_2;\, J_2J'')R(JL_2J'L_1;\, J''L) \tag{8.2}$$

which is called the 9-j symbol of the OSP(1, 2) algebra, while

$$G=n(L_1L_2L)+2(L_1+L_2+L)\lambda(2L)+\lambda(2L_1)\lambda(2L_2)+n(J_1JJ_2)$$
$$+[2(J_1+J_2+J)+\lambda(2J'_2)]\lambda(2J_1)+[1+\lambda(2J')]\lambda(2J'_1).$$

The following is a particular case of the formula (8.1):

$$(2J'_1, 2J'_2, 2J\|[\boldsymbol{S}^{2L}(1)\times\boldsymbol{T}^{2L}(2)]^{0}\|2J_1, 2J_2, 2J)$$
$$=(-1)^{G_0}(2J'_1\|\boldsymbol{S}^{2L}(1)\|2J_1)(2J'_2\|\boldsymbol{T}^{2L}(2)\|2J_2)R(J_1LJJ'_2;\, J'_1J_2) \tag{8.3}$$

where

$$G_0=\lambda(2L)+\lambda(2J'_2)[\lambda(2J_2)+1].$$

In deriving the result (8.3), we have used a particular value for the 9-j symbol†

$$\begin{pmatrix}J_1 & J_2 & J\\ L & L & 0\\ J'_1 & J'_2 & J\end{pmatrix}=(-1)^{h}R(J_1LJJ'_2;\, J'_1J_2) \tag{8.4}$$

where

$$h=n(J_1JJ_2)+2L+2(J_1+J_2+J)[\lambda(2J_1)+\lambda(2J'_2)]+2(J'_1+J'_2+J)[1+\lambda(2J)].$$

† The relation (8.4) is obtained from the sum rules of the OSP(1, 2) Racah coefficients.

The reader may check the correctness of the following result derived from the formula (8.3):

$$(2J\|[\boldsymbol{q}^2(1)\times\boldsymbol{q}^2(2)]^0\|2J)=(2J_1\|\boldsymbol{q}^2(1)\|2J_1)(2J_2\|\boldsymbol{q}^2(2)\|2J_2)R(J_1 1 J J_2;\ J_1 J_2). \tag{8.5}$$

9. Conclusion

We have established a complete theory on tensor operators of the OSP(1, 2) algebra, in which all the questions associated with the calculations of matrix elements of tensor operators have been solved.

The use of an indefinite metric is a fundamental factor leading us to this success. It is important that we need to distinguish between the metrics of coupling spaces and those of in-coupling spaces, and identify the different ways of coupling which generate vectors with the same quantum numbers J and M.

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